J. Fluid Mech. (1966), vol. 26, part 3, pp. 599–606
Printed in Great Britain

Periodic oscillations in a model of thermal convection

By JOSEPH B. KELLER†
Courant Institute of Mathematical Sciences,
New York University, New York, N.Y.

(Received 4 May 1966)

Periodic oscillations are found in a one-dimensional model of thermal convection. The model consists of a fluid-filled tube bent into rectangular shape and standing in a vertical plane. The fluid is heated at the centre of the lower horizontal segment and cooled at the centre of the upper horizontal segment. When a certain parameter exceeds unity, a periodic motion of the fluid is found in which the flow is always in the same direction but in which the speed varies. Inertia is unimportant for this oscillation, which depends upon the interplay between frictional and buoyancy forces.

1. Introduction

To study thermal convection, Welander (1965) proposed a simple one-dimensional model consisting of a fluid-filled tube bent into rectangular shape and standing in a vertical plane. The fluid is heated at the centre of the lower horizontal segment and cooled at the centre of the upper horizontal segment. In experimenting with this model, he found that the fluid performed a few oscillations before settling down to steady convection. It is our purpose to study these oscillations.

Our main result is that, under suitable conditions, the model exhibits periodic motions. It is a self-excited oscillator. In these periodic motions the fluid always moves in the same direction and its speed varies periodically. Furthermore, the oscillations can occur in the absence of inertial effects, merely requiring an interplay between frictional and buoyancy forces. These facts will be demonstrated by finding an exact explicit solution of the non-linear equations governing the motion of the fluid in the model. The existence of similar oscillations in Bénard cells is suggested by T. Rossby's motion pictures of such cells at high Rayleigh number.

The present model has a state of rest, a state of steady convection provided a certain parameter is less than one, and a periodic motion if that parameter exceeds one. The fact that there is no steady state in the latter case is a defect of our simplified (discontinuous) characterization of the heating and cooling elements. For a more realistic (continuous) characterization we show that there is always a steady state. We also show that this steady state is unstable under certain conditions. Then presumably periodic motions will occur for the continuous model, just as for our discontinuous one.

† Part of this work was done while the author was a member of the Geophysical Fluid-dynamics Project, Woods Hole Oceanographic Institute, Woods Hole, Mass.

Welander (1965) has made a detailed study of the stability of the steady state for a particular continuous model. For that model he has also solved the initial-value problem numerically. It appears that oscillations occur when the steady state is unstable, as we should expect.

2. Formulation

Let us consider two vertical pipes of equal length joined together at the top and bottom by two horizontal pipes (see figure 1). We assume that there is a heating element at the mid-point of the lower pipe and a cooling element at the mid-point of the upper pipe. The pipes are supposed to contain a fluid with volume coefficient of thermal expansion α. Thus the density ρ of the fluid at

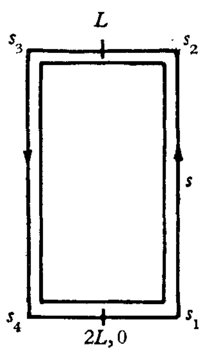

FIGURE 1. A tube bent into rectangular form. Arc length s is measured counter-clockwise from the heating element at the mid-point of the lower horizontal segment. The corners are at s_1, $s_2 = L-s_1$, $s_3 = L+s_1$ and $s_4 = 2L-s_1$. The cooling element is at $s = L$.

temperature T is assumed to be $\rho = \rho_0(1+\alpha T)$. As a consequence of temperature variations in the fluid there will be density variations. They may result in a net buoyancy force on the fluid due to gravity. Such a buoyancy force will produce a motion of the fluid, which we wish to investigate.

In order to describe the temperature distribution we introduce the arc length s measured counter clockwise along the pipes from the heating element. Let the corners be at s_1, $s_2 = L-s_1$, $s_3 = L+s_1$ and $s_4 = 2L-s_1$, the cooling element at $s = L$ and the heating element at $s = 0$, which is the same as $s = 2L$. We denote the temperature at position s and time t by $T(s,t)$ and the counter clockwise velocity of the fluid, assumed to be the same at all points of the fluid, by $u(t)$. If A denotes the constant cross-sectional area of the pipes and $-\mu u$ denotes the frictional force on the fluid, the equation of motion for the fluid is

$$2LA\rho_0 u_t+\mu u = \alpha\rho_0 Ag\left[\int_{s_1}^{s_2} T(s,t)\,ds-\int_{s_3}^{s_4} T(s,t)\,ds\right]. \tag{2.1}$$

To determine the temperature we assume that thermal conduction is negligible compared with thermal convection. Therefore T satisfies the equation

$$T_t+uT_s = 0. \tag{2.2}$$

We also assume that the heating and cooling elements are characterized by relations which determine the temperature of the fluid coming out of an element in terms of the temperature of the fluid entering that element. These relations will involve the velocity of the fluid. We take them to be

$$T(0\pm,t) = [T_h - T(0\mp,t)]f(u/u_0), \tag{2.3}$$

$$T(L\pm,t) = T_c - [T_c - T(L\mp,t)]f(u/u_0). \tag{2.4}$$

In these relations $\pm$ denotes the sign of u so that T on the left side of each equation denotes the temperature of the fluid coming out of the heating or cooling element, while T on the right side denotes the temperature of the fluid entering the corresponding element. The temperatures T_h and T_c are supposed to be the temperatures of the fluid leaving the elements when $|u|$ is small compared with the characteristic velocity u_0. On the other hand, when $|u|$ is large compared with u_0, the temperature of the fluid should be unchanged by the elements. Therefore the function $f(u/u_0)$ must increase monotonically from zero to unity as $|u|/u_0$ increases from zero to infinity. In addition it must be an even function of u.

The problem we consider is that of solving equations (2.1) –(2.4) for $u(t)$ and $T(s,t)$. In the initial-value problem we assume that $u(0)$ and $T(s,0)$ are given, while in the oscillation problem we seek periodic solutions. The function $f(u/u_0)$ must also be given.

3. New variables

It is convenient to introduce the dimensionless variables s', t', u' and T', defined by

$$\left.\begin{aligned} s' = s/L, \quad t' = tu_0/L, \quad u' = u/u_0, \\ T'(s',t') = [T(s,t) - \tfrac{1}{2}(T_h + T_c)]/\tfrac{1}{2}(T_h - T_c). \end{aligned}\right\} \tag{3.1}$$

In terms of these variables (2.1)–(2.4) become the following equations, from which all primes have been omitted,

$$2\beta u_t + 2\gamma u = \int_{s_1}^{s_2} T(s,t)\,ds - \int_{s_3}^{s_4} T(s,t)\,ds, \tag{3.2}$$

$$T_t + uT_s = 0, \tag{3.3}$$

$$T(0\pm,t) = 1 + [T(0\mp,t) - 1]f(u), \tag{3.4}$$

$$T(1\pm,t) = -1 + [T(1\mp,t) + 1]f(u). \tag{3.5}$$

Here β and γ are defined by

$$\beta = 2u_0^2/\alpha g L(T_h - T_c), \tag{3.6}$$

$$\gamma = \mu u_0/\alpha g \rho_0 A L(T_h - T_c). \tag{3.7}$$

Now s ranges from 0 to 2 and the cooling element is at $s = 1$.

We shall restrict our attention to temperature distributions which are of the antisymmetric form

$$T(s,t) = -T(1+s,t). \tag{3.8}$$

For such distributions we see that (3.5) is an immediate consequence of (3.4) and that (3.2) reduces to

$$\beta u_t + \gamma u = \int_{s_1}^{s_2} T(s,t)\,ds. \tag{3.9}$$

Thus we need merely determine $u(t)$ and $T(s,t)$ for $0 \leqslant s \leqslant 1$ satisfying (3.3), (3.4) and (3.9). In (3.4) $T(0-,t)$ occurs, but by (3.8) $T(0-,t) = -T(1-,t)$.

To solve (3.3) let us introduce x, defined by

$$x(t) = \int_0^t u\,dt. \tag{3.10}$$

Then the solution of (3) may be written as

$$T(s,t) = T(x-s) \quad (0 \leqslant s \leqslant 1). \tag{3.11}$$

If we consider u to be a function of x then $u_t = uu_x$ and (3.9) becomes

$$\beta uu_x + \gamma u = \int_{x-s_2}^{x-s_1} T(x')\,dx'. \tag{3.12}$$

By using (3.11) we can write (3.4) as

$$T(x) = 1-[T(x-1)+1]f(u) \quad (u > 0), \tag{3.13}$$

$$T(x-1) = -1+[1-T(x)]f(u) \quad (u < 0). \tag{3.14}$$

Thus the problem is reduced to solving (3.12)–(3.14) for $u(x)$ and $T(x)$. In the initial-value problem we must be given $u(0)$ and $T(x)$ for $-1 \leqslant x \leqslant 0$. For a solution in which $u(x) > 0$, we need solve only (3.12) and (3.13).

4. Steady-state solution

Let us first seek a steady or time-independent solution with $u \neq 0$. If $T_t = 0$ then it follows from (3.3) that $T_s = 0$ so T is a constant in the interval $0 \leqslant s \leqslant 1$. Then (3.9) becomes

$$u = T(s_2-s_1)\gamma^{-1}. \tag{4.1}$$

From (3.4) we obtain two equations, one for $u > 0$ and one for $u < 0$. They are

$$(1-T)/(1+T) = f(u) \quad (u > 0), \tag{4.2}$$

$$(1+T)/(1-T) = f(u) \quad (u < 0). \tag{4.3}$$

Eliminating u between (4.1) and (4.2) yields

$$(1-T)/(1+T) = f[T(s_2-s_1)\gamma^{-1}]. \tag{4.4}$$

The left side of (4.4) decreases monotonically from $+\infty$ at $T = -1$ to zero at $T = +1$. The right side is positive, and monotonically increasing from zero to one as T increases from 0 to ∞. Therefore (4) has exactly one solution $T_1 > 0$, provided f is continuous. The corresponding velocity $u_1 = T_1(s_2-s_1)\gamma^{-1}$ given by (4.1) is also positive. Equations (4.1) and (4.3) become identical with (4.1) and (4.2) when u and T are replaced by $-u$ and $-T$. Therefore $-T_1$ is the unique solution of (4.1) and (4.3) and the corresponding velocity $-u_1$ is negative.

If $u = 0$ then (3.3) does not imply that T is independent of s in a steady state. In addition (3.4) does not apply. Then (3.9) yields the sole equilibrium condition

$$\int_{s_1}^{s_2} T(s)\,ds = 0. \tag{4.5}$$

There are obviously infinitely many equilibrium distributions satisfying (4.5).

5. Periodic solution for an inertialess system

We shall now obtain a periodic solution of (3.12)–(3.14) by setting $\beta = 0$, which means omitting inertial effects. In addition we shall choose for $f(u)$ the discontinuous function

$$\left.\begin{aligned} f(u) &= 0 \quad (|u| < 1),\\ f(u) &= 1 \quad (|u| > 1).\end{aligned}\right\} \tag{5.1}$$

To construct the solution we consider the initial-value problem with $T(x)$ given by

$$T(x) = 1 \quad (-1 \leqslant x \leqslant 0). \tag{5.2}$$

When $\beta = 0$ it is not necessary to specify $u(0)$. We shall show that the solution of (3.12)–(3.14) with $\beta = 0$, f given by (5.1) and T given initially by (5.2) is periodic. The solution can be found by drawing a picture of the cell and following the motion of the hot and cold fluid, however, we shall give an analytic derivation.

When $\beta = 0$, (3.12) becomes

$$u(x) = \gamma^{-1}\int_{x-s_2}^{x-s_1} T(x')\,dx'. \tag{5.3}$$

With f given by (5.1), (3.13) becomes

$$T(x) = 1 \quad (0 < u(x) < 1), \tag{5.4}$$

$$T(x) = -T(x-1) \quad (1 < u(x)). \tag{5.5}$$

We shall not need (3.14) since u will always be positive. From (5.2) and (5.3) we obtain

$$u(x) = u_1 \quad (0 < x < s_1), \tag{5.6}$$

where u_1 is defined by

$$u_1 = \gamma^{-1}(s_2 - s_1). \tag{5.7}$$

We shall assume that $u_1 > 1$, which implies that

$$\gamma < 1 - 2s_1 \tag{5.8}$$

From (5.6) and our assumption (5.8) it follows that $u(x) > 1$ for $0 < x < s_1$. Let x_2 be the smallest positive value of x for which $u(x) = 1$. Then (5.5) yields

$$T(x) = -1 \quad (0 < x < x_2). \tag{5.9}$$

Now (5.9) and (5.2) can be used in (5.3) to yield

$$u(x) = u_1 - 2\gamma^{-1}(x - s_1) \quad (s_1 \leqslant x \leqslant x_2 + s_1). \tag{5.10}$$

To find x_2 we set $u(x_2) = 1$ in (5.10) and obtain

$$x_2 = \tfrac{1}{2}(1-\gamma). \tag{5.11}$$

Let x_3 be the smallest value of $x > x_2$ for which $u(x) = 1$. Then from (5.4) we have

$$T(x) = 1 \quad (x_2 < x < x_3). \tag{5.12}$$

Now (5.12) can be used in (5.3), together with (5.9) and (5.2), to yield

$$u(x) = 1 - 2\gamma^{-1}s_1 \quad (x_2 + s_1 \leqslant x \leqslant s_2), \tag{5.13}$$

$$u(x) = 1 + 2\gamma^{-1}(x-1) \quad (s_2 \leqslant x \leqslant x_3). \tag{5.14}$$

To find x_3 we set $u(x_3) = 1$ in (5.14) and obtain

$$x_3 = 1. \tag{5.15}$$

We now use (5.9) in (5.5) to obtain

$$T(x) = 1 \quad (1 < x < 1+x_2). \tag{5.16}$$

By using (5.16) and (5.12) in (5.3) we find that (5.14) holds for

$$x_2 \leqslant x \leqslant 1+\tfrac{1}{2}(s_2-s_1-\gamma)$$

and

$$u(x) = u_1 \quad (1+\tfrac{1}{2}(s_2-s_1-\gamma) \leqslant x \leqslant 1+x_2). \tag{5.17}$$

Since $u(x) > 1$ for $1 < x < 1+x_2$, our use of (5.9) in deriving (5.16) is justified. The fact that by (5.12) and (5.16) $T(x) = 1$ in the interval $x_2 < x < 1+x_2$ shows that $T(x)$ is periodic with period $1+x_2$.

In the above analysis we must require that $u(x)$ be positive in order that the introduction of x as a new variable be valid. Therefore the minimum velocity (5.13) must be positive, which implies

$$\gamma > 2s_1. \tag{5.18}$$

This restriction and (5.8) yield $2s_1 < \gamma < 1-2s_1$, which implies that $s_1 < \frac{1}{4}$.

To reintroduce the time t instead of the distance x we solve the equation $dx/dt = u(x)$ with the initial condition $x(0) = 0$ to obtain

$$t = \int_0^x \frac{dx}{u(x)}. \tag{5.19}$$

By using the preceding results for $u(x)$ in (5.19) we obtain

$$x = u_1 t \quad (0 \leqslant t \leqslant s_1/u_1), \tag{5.20}$$

$$x = s_1+\frac{\gamma u_1}{2}\left[1-\exp\left\{-2\gamma^{-1}\left(t-\frac{s_1}{u_1}\right)\right\}\right] \quad \left(s_1/u_1 \leqslant t \leqslant \frac{s_1}{u_1}+\tfrac{1}{2}\gamma\log\frac{1-2s_1}{\gamma-2s_1}\right), \tag{5.21}$$

$$x = (1-2\gamma^{-1}s_1)\left(t-\frac{s_1}{u_1}-\tfrac{1}{2}\gamma\log\frac{1-2s_1}{\gamma-2s_1}\right)+\frac{1-\gamma}{2}+s_1$$
$$\left(\frac{s_1}{u_1}+\frac{\gamma}{2}\log\frac{1-2s_1}{\gamma-2s_1} \leqslant t \leqslant t'\right). \tag{5.22}$$

Here t' is defined by

$$t' = \frac{s_1}{u_1}+\frac{\gamma}{2}\log\frac{1-2s_1}{\gamma-2s_1}+\frac{1+\gamma-4s_1}{2-4\gamma^{-1}s_1}, \tag{5.23}$$

$$x = 1-\tfrac{1}{2}\gamma+(\tfrac{1}{2}\gamma-s_1)\exp\{2\gamma^{-1}(t-t')\}, \quad t' \leqslant t \leqslant t'+\tfrac{1}{2}\gamma\log[(1-2s_1)/(\gamma-2s_1)], \tag{5.24}$$

$$x = \tfrac{1}{2}(3-\gamma)-s_1+u_1(t-t')-\tfrac{1}{2}\gamma\log[(1-2s_1)/(\gamma-2s_1)],$$
$$t'+\tfrac{1}{2}\gamma\log\frac{1-2s_1}{\gamma-2s_1} \leqslant t \leqslant t'+\tfrac{1}{2}\gamma\log\frac{1-2s_1}{\gamma-2s_1}+\frac{s_1}{u_1}. \tag{5.25}$$

The upper limit in (5.25) is the period P of the motion. From (5.23) and (5.25) it is

$$P = \frac{2\gamma s_1}{1-2s_1}+\gamma\log\frac{1-2s_1}{\gamma-2s_1}+\frac{1+\gamma-4s_1}{2-4\gamma^{-1}s_1}. \tag{5.26}$$

Graphs of $T(x)$ and $u(x)$ are shown in figure 2.

6. Stability of the steady motion

The discontinuous function $f(u)$ employed in the preceding section is such that with it the cell has no steady motion, provided (5.8) is satisfied. In §4 we showed that there is always one pair of steady motions when $f(u)$ is continuous. This might suggest that the periodic motion we have just found occurs because we chose a discontinuous f. We shall now show that this is not the case by proving

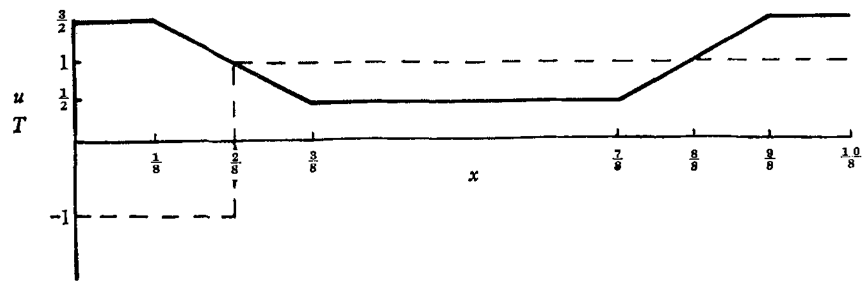

FIGURE 2. Graphs of u (solid curve) and T (broken curve) as functions of x for one period of the periodic solution found in §5. For this graph $s_1 = \frac{1}{8}$, $\gamma = \frac{1}{2}$, $u_1 = \frac{3}{2}$, $x_2 = \frac{1}{4}$, the period is $1\frac{1}{4}$ and the minimum velocity is $\frac{1}{2}$.

that, if we make f continuous by introducing a steep linear section, the resulting steady solution is unstable. This result shows that, if there is a stable solution for such an f, it is not steady. Our previous solution suggests that it is periodic.

Let $f(u) = 0$ for $|u| \leqslant 1, f(u) = 1$ for $|u| \geqslant 1+\delta$ and

$$f(u) = \delta^{-1}(u-1) \quad (1 \leqslant u \leqslant 1+\delta). \tag{6.1}$$

Then for $0 < \delta \ll 1$, the solution u_s, T_s of the steady-motion equation (4.4) is

$$u_s = 1+\delta[(u_1-1)/(u_1+1)]+O(\delta^2), \quad T_s = u_s/u_1. \tag{6.2}$$

The variational equations of (3.12) and (3.13), obtained by differentiating these equations with respect to some unspecified parameter, are at $u = u_s$, $T = T_s$,

$$\beta u_s \dot{u}_x + \gamma \dot{u} = \int_{x-s_2}^{x-x_1} \dot{T}\, dx, \tag{6.3}$$

$$\dot{T}(x) = -\dot{T}(x-1) f(u_s) - f'(u_s)(T_s+1)\,\dot{u}. \tag{6.4}$$

Here $\dot{T}$ and $\dot{u}$ denote the variations of T and u, i.e. their derivatives with respect to a parameter.

We shall seek a solution of (6.3) and (6.4) of the form

$$\dot{u} = U e^{i\omega x}, \tag{6.5}$$

$$\dot{T} = V e^{i\omega x}. \tag{6.6}$$

Using (6.5) and (6.6) in (6.3) and (6.4) yields

$$(i\omega\beta u_s + \gamma)\, U = (V/i\omega)(e^{-i\omega s_1} - e^{-i\omega s_2}), \tag{6.7}$$

$$V = -V f(u_s)\, e^{-i\omega} - f'(u_s)(T_s+1)\, U. \tag{6.8}$$

Eliminating V/U yields

$$i\omega(i\omega\beta u_s+\gamma)\,[1+f(u_s)\,e^{-i\omega}]+f'(u_s)\,(T_s+1)\,(e^{-i\omega s_1}-e^{-i\omega s_2}) = 0. \qquad (6.9)$$

Inserting the values of u_s and T_s yields

$$(e^{-i\omega s_1}-e^{-i\omega s_2})\left(\frac{1}{u_1}+\frac{\delta}{u_1}\frac{u_1-1}{u_1+1}\right)+\delta(-\omega^2\beta+i\omega\gamma)\left(1+\frac{u_1-1}{u_1+1}e^{-i\omega}\right)+O(\delta^2) = 0. \qquad (6.10)$$

From (6.10) we see that ω is a function of δ, $\omega(\delta)$. For $\delta = 0$, (6.10) yields

$$e^{-i\omega(0)s_1}-e^{-i\omega(0)s_2} = 0. \qquad (6.11)$$

Thus, since $s_2-s_1 = 1-2s_1$ the solution of (6.11) may be written as

$$\omega(0) = \frac{2\pi n}{1-2s_1} \quad (n = 0, \pm 1, \pm 2, \ldots). \qquad (6.12)$$

Differentiating (6.10) with respect to δ at $\delta = 0$ yields

$$(-i\omega' s_1 e^{-i\omega s_1}+i\omega' s_2 e^{-i\omega s_2})\frac{1}{u_1}+(-\omega^2\beta+i\omega\gamma)\left(1+\frac{u_1-1}{u_1+1}e^{-i\omega}\right) = 0. \qquad (6.13)$$

Solving for ω' leads to

$$\omega'(0) = \frac{-u_1}{1-2s_1}(\omega\gamma+i\omega^2\beta)\left(e^{-i\omega s_1}+\frac{u_1-1}{u_1+1}e^{-i\omega(1+s_1)}\right). \qquad (6.14)$$

Now $\omega(\delta) = \omega(0)+\delta\omega'(0)+O(\delta^2)$. Thus

$$\begin{aligned}\operatorname{Im}\omega(\delta) &= \delta \operatorname{Im}\omega'(0)+O(\delta^2)\\ &= \frac{\delta u_1}{1-2s_1}\left[\omega\gamma \sin\omega s_1+\frac{u_1-1}{u_1+1}\sin\omega(1+s_1)-\omega^2\beta\right.\\ &\qquad \left.\times\left(\cos\omega s_1+\frac{u_1-1}{u_1+1}\cos\omega(1+s_1)\right)\right]+O(\delta^2). \qquad (6.15)\end{aligned}$$

On the right side in (6.15), ω denotes $\omega(0)$, which is given by (6.12). It is clear that for any choice of the parameters γ, s_1 and β (provided s_1 and β are not both zero) there are values of n in (6.12) for which the right side of (6.15) is negative. Therefore the steady state is unstable when δ is sufficiently small. This confirms the assertion in the first paragraph of this section.

Support for this project was given by the U.S. Army Research Office, Durham, under contract no. DA-31-124-ARO-D-361.

REFERENCE

WELANDER, P. 1965 Steady and oscillatory motions of a differentially heated fluid loop, Woods Hole Oceanographic Institute. Ref. no. 65–48 (unpublished manuscript).